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ON THE FORCE ACTING ON A CYLINDER IN A STEADY STREAM OF VISCOUS FLUID AT LOW REYNOLDS NUMBER*

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The plane flow over a circular cylinder of a steady stream of viscous fluid at low Reynolds numbers is considered. A rigorous derivation of Lamb's formula for drag is given with an estimate of the residual term.

1. The flow over a cylindrical body of a plane-parallel steady stream of viscous incompressible fluid is defined by the system of Navier-Stokes equations with boundary conditions

$$\Delta v - 2\lambda (u_\infty \cdot \nabla) v - 2\lambda \operatorname{grad} p = 2\lambda \sum_{k=1}^{2} v_k \frac{\partial v}{\partial x_k}, \quad \operatorname{div} v = 0 \tag{1.1}$$

$$v|_C = -u_\infty, \quad \lim_{|x|\to\infty} v(x) = 0 \tag{1.2}$$

where $v = u - u_\infty$, u and p are the dimensionless velocity vector and pressure, respectively, 2λ is the Reynolds number, u_∞ is the vector of the oncoming stream velocity, $x = (x_1, x_2)$, and C is the contour of the transverse cross section B of the body in the stream. We assume the coordinate origin to be inside contour C with the coordinate axes directed so that $u_\infty = (1, 0)$.

Linear Oseen equations

$$\Delta v - 2\lambda (u_\infty \cdot \nabla) v - 2\lambda \operatorname{grad} p = f(x), \quad \operatorname{div} v = 0 \tag{1.3}$$

are used as an auxilliary system in the investigation of the boundary value problem (1.1),(1.2), whose solution can be represented in the form of series /1/

$$v(x, \lambda) = v^{(0)}(x, \lambda) + \sum_{k=1}^{\infty} v^{(k)}(x, \lambda)(2\lambda)^k \tag{1.4}$$

that is convergent for reasonably low Reynolds numbers. In formula (1.4) $v^{(0)}(x, \lambda)$ represents the solution of the homogeneous system of Oseen equations (with $f(x) = 0$) with boundary conditions (1.2), and $v^{(l)}(x, \lambda)$ $(l \geqslant 1)$ is the solution of the inhomogeneous system (1.3) for

$$f(x) = \sum_{k=1}^{2} \sum_{j=0}^{l-1} v_k^{(j)} \frac{\partial v^{(l-1-j)}}{\partial x_k}$$

with null boundary conditions $v|_C = 0$, $\lim v = 0$ $(|x| \to \infty)$.

2. The formula for drag of a body in a steady three-dimensional stream of a viscous incompressible fluid had been obtained earlier (**). Using similar reasoning it is possible to obtain that formula also for a two-dimensional plane flow

$$F_1 = F_1^{(0)} - \int_D w_j^{(0)} v_k \frac{\partial v_j}{\partial y_k} dy \tag{2.1}$$

*Prikl.Matem.Mekhan.,45,pp.845-848,1981

**) K.I. Babenko, The theory of perturbations of steady flows of viscous incompressible fluid at low Reynolds numbers. Preprint No.79, Inst. Prikl.Matem., Akad. Nauk SSSR, 1975.

where $F_1^{(0)}$ is the drag in the Oseen approximation when $D = R^2 \setminus B$, $w^{(0)}$ is the velocity of perturbations in that approximation when $u_\infty = (-1, 0)$. It is assumed that summation from 1 to 2 is carried out over twice recurrent subscripts.

Formulas (2.1) are obtained using the integral representation of solution and some of its estimates which appear in /2/ and, also, the readily verified equality

$$\int_{C_R} H_{ij}(x-y)\, n_1(x)\, dl_x = -\frac{\delta_{ij}}{4\pi} + o(1)$$

where C_R is a circle of radius R ($R \to \infty$) and H_{ij} is the fundamental solution of the Oseen equation

$$H_{11} = \frac{Q+T_1}{4\pi}, \quad H_{22} = \frac{Q-T_1}{4\pi}, \quad H_{12} = H_{21} = \frac{T_2}{4\pi}$$

$$T_k = \frac{y_k - x_k}{|x-y|}\left[\frac{1}{\lambda|x-y|} - K_1(\lambda|x-y|)\, e^{\lambda(x_1-y_1)}\right] \quad (k = 1, 2)$$

$$Q = K_0(\lambda\,|\,x-y\,|)e^{\lambda(x_1-y_1)}$$

where K_0, K_1 are MacDonald functions.

3. Formula (2.1) enables us to obtain an asymptotic formula for the determination of drag of a circular cylinder in the case of low Reynolds number. The drag $F_1^{(0)}$ of a cylinder appearing in that formula in the Oseen approximation was investigated in /3,4/ and other works. Solution of the problem of flow over a circular cylinder was obtained in /3/ in polar coordinates r, θ in the form

$$v_r^{(0)} = -\sum_{n=0}^{\infty} A_n \frac{\cos n\theta}{r^{n+1}} - \frac{1}{4}\sum_{m=0}^{\infty} B_m\left[\frac{2}{\xi} + \sum_{n=1}^{\infty}\Phi_{mn}(\xi)\cos n\theta\right] \tag{3.1}$$

$$v_\theta^{(0)} = -\sum_{n=1}^{\infty} A_n \frac{\sin n\theta}{r^{n+1}} - \frac{1}{4}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_m \Psi_{mn}(\xi)\sin n\theta \tag{3.2}$$

$$\Phi_{mn} = (K_{m+1} + K_{m-1})(I_{m-n} + I_{m+n}) + K_m\,(I_{m-n-1} + I_{m-n+1} + I_{m+n-1} + I_{m+n+1})$$

$$\Psi_{mn} = (K_{m+1} - K_{m-1})(I_{m-n} - I_{m+n}) + K_m\,(I_{m-n-1} - I_{m-n+1} - I_{m+n-1} + I_{m+n+1})$$

where I_m, K_m are modified Bessel functions of arguments $\xi = \lambda r$, and A_n, B_m are constants.

Boundary conditions on the body (at $r = 1$) $v_r = -\cos\theta$, $v_\theta = \sin\theta$ are satisfied when

$$A_0 + \frac{1}{2\lambda}\sum_{m=0}^{\infty} B_m = 0; \quad A_n + \frac{1}{4}\sum_{m=0}^{\infty} B_m\Phi_{mn}(\lambda) = \delta_{n1}$$

$$A_n + \frac{1}{4}\sum_{m=0}^{\infty} B_m\Psi_{mn}(\lambda) = -\delta_{n1} \quad (n = 1, 2, \ldots)$$

Eliminating A_n we can obtain the following system of equations for the determination of coefficients B_m:

$$\sum_{m=0}^{\infty} B_m\Lambda_{m,n}(\lambda) = 4\delta_{n1} \quad (n = 1, 2, \ldots) \tag{3.3}$$

$$\Lambda_{m,n} = I_{m-n}K_{m-1} + I_{m+n}K_{m+1} + K_m\,(I_{m-n+1} + I_{m+n-1}) \tag{3.4}$$

Numerical investigation of the behavior of coefficients B_m carried out in /3/ at some Reynolds number had shown that these coefficients rapidly decrease in absolute value, as m is increased. Calculations have, also, shown that coefficient B_0 decreases as the Reynolds number is lowered. The formulas presented in /3/ were limited to a single coefficient B_0.

The method proposed by K.I. Babenko is used below for analyzing the solution of Eqs.(3.3). Setting $C_m = B_{m-1}\Lambda_{m-1,m}$, $\mu_{nm} = \Lambda_{m-1,n}/\Lambda_{m-1,m}$ we obtain for the determination of $C_1, C_2, \ldots$ the system of equations

$$\sum_{m=1}^{\infty} \mu_{nm}C_m = 4\delta_{n1} \quad (n = 1, 2, \ldots) \tag{3.5}$$

with diagonal elements μ_{nn} equal unity. We denote the remainder of matrices $M = (\mu_{nm})$ and the unit matrix $E = (\delta_{nm})$ by N, and rewrite the system of Eqs.(3.5) in the form

$$(E + N)C = f \qquad (3.6)$$
$$C = (C_1, C_2, \ldots)', \quad f = (4, 0, 0, \ldots)'$$

where the prime indicates transposition.

It is possible to show that the elements of matrix $N = (\nu_{nm})$ satisfy for $(n, m) \neq (1, 2)$ the inequality $\nu_{nm} < CI_{|n-m|}(\lambda)$, and $\nu_{12} = 2K_0(\lambda)I_1(\lambda) + O(\lambda)$. This and Eq.(3.6) imply that for fairly small λ

$$C = (E + N)^{-1}f = f - Nf + N^2f - \ldots = f + O(\lambda S) \qquad (3.7)$$
$$S = {}^1/_2 - \gamma - \ln(\lambda / 2)$$

where $\gamma \approx 0.57721$ is the Euler constant.

The following formula was obtained in /5/ for the drag of a cylinder:

$$F_1 = -\sqrt{\frac{2\pi}{\lambda}} \lim_{r\to\infty} (\sqrt{r} v_r)\Big|_{\theta=0}$$

The substitution of expression (3.1) for v_r yields

$$F_1^{(0)} = -\sqrt{\frac{2\pi}{\lambda}} \lim_{r\to\infty} (\sqrt{r} v_r^{(0)})\Big|_{\theta=0} = \frac{\pi}{\lambda}\sum_{m=0}^{\infty} B_m = \frac{\pi}{\lambda}\sum_{m=1}^{\infty} \frac{C_m}{\Lambda_{m-1,\,m}}$$

From this and formulas (3.4) and (3.7) follows that

$$F_1^{(0)} = \frac{2\pi}{\lambda\,[I_0(\lambda)\,K_0(\lambda) + I_1(\lambda)\,K_1(\lambda)]} + O(\lambda S)$$

Hence formula (2.3) may be written as

$$F_1 = \frac{2\pi}{\lambda\,(I_0K_0 + I_1K_1)} - \int_D w_j^{(0)} v_k \frac{\partial v_j}{\partial y_k}\,dy + O(\lambda S) \qquad (3.8)$$

4. To evaluate the integral

$$J = -\int_D w_j^{(0)} v_k \frac{\partial v_j}{\partial y_k}\,dy = \int_D v_j v_k \frac{\partial w_j^{(0)}}{\partial y_k}\,dy$$

which appears in formula (3.8), we can use the estimates v_j and $\partial v_j^{(0)}/\partial y_k$, as $\lambda \to 0$ in /1,6/ which imply that

$$|J| < C\lambda^{-1}\ln^{-3}(1/\lambda)$$

For the drag of a circular cylinder at low Reynolds number we, thus, obtain the following expression:

$$F_1 = \frac{2\pi}{\lambda\,[I_0(\lambda)\,K_0(\lambda) + I_1(\lambda)\,K_1(\lambda)]} + O\left(\frac{1}{\lambda}\ln^{-3}\frac{1}{\lambda}\right) \qquad (4.1)$$

This formula appeared in /7/ without an estimate of the residual term. Separating in (4.1) the principal term, we obtain Lamb's formula /8/ with the estimate of the residue

$$F_1 = \frac{2\pi}{\lambda S} + O\left(\frac{1}{\lambda}\ln^{-3}\frac{1}{\lambda}\right) \qquad (4.2)$$

Formulas (4.1) and (4.2) for the drag of a cylinder correspond to formulas obtained in /9/ by the method of merging asymptotic expansions.

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